Pion-Deuteron Elastic Scattering at 142 MeV and the Form-Factor Approximation*†

HUGH N. PENDLETON
Brandeis University, Waltham, Massachusetts
(Received 19 November 1962)

The differential cross section for charged pions with laboratory kinetic energy of 142 MeV to be scattered elastically by deuterons is calculated in a form-factor approximation which includes double scattering effects. The calculations also include effects produced by the D-state part of the deuteron wave function and the repulsive core in the neutron-proton interaction. A meson-theory derivation of an impulse- approximation series for the scattering amplitude using the Heitler-London method is described, as is an "optimal" procedure for obtaining a form-factor approximation to that series. The differences between the optimal form-factor approximation and other versions of the impulse approximation are discussed. The comparison of the calculations with experimental data indicates that pion-deuteron elastic scattering in the region of the first pion-nucleon resonance depends on off-energy-shell features of the pion-nucleon interaction which cannot be treated by a simple form-factor approximation.

I. INTRODUCTION

FOR several reasons the elastic scattering of π^- mesons by deuterons provides a good test of our understanding of the implications of a bound-state model applied to the scattering of strongly interacting particles. First of all, there is little doubt that a deuteron is a bound state of a neutron and a proton with a wave function whose properties are fairly well known.[1] Second, the weak binding of the deuteron and the large distance between its components encourage one to believe that a relatively simple approximate expression can be given for the scattering amplitude. Such an expression, which will be some version of the impulse approximation,[2] will require knowledge of the pion-nucleon scattering amplitudes; fortunately, a large amount of information about those amplitudes exists.[3] Third, any theoretical expression for the elastic pion-deuteron scattering can be compared with experiment to check our understanding since experimental data in the form of differential cross sections for pions with laboratory energies of 61,[4] 85,[5] 140,[6] 142,[7] and 300 MeV[8] are available.

If such a check shows that pion-deuteron scattering is understood, an attempt might be made to transfer pion-deuteron methods to more difficult problems such as $\pi-\Sigma$ scattering or $\pi-\Lambda$ scattering on the assumption that Σ's and Λ's are bound states.[9] One might then have a detailed quantitative criterion for deciding whether or not it is useful to regard some particles as compounds of others. Such a criterion would be a desirable supplement to those bound-state models which have a certain popularity in elementary particle physics.[9,10] If the experimental check shows that pion-deuteron scattering is not understood (as, in fact, is the case), it clearly indicates that the impulse approximation expressions must be improved until understanding is reached before one can hope to apply impulse-approximation expressions successfully to more complicated bound-state problems.

If particle X is a bound state of Y and Z, one may hope to express the scattering of W on X in terms of scattering parameters of the WY and WZ systems and wave-function parameters of the YZ system. This hope is realized in various impulse approximations[2,11–13] to the impulse series,[14] several of which have been applied to pion-deuteron scattering.[4,11,12,15] The simplest impulse expression says that

$$T(W,X)=K[T(W,Y)+T(W,Z)]\times\int\exp(i\mathbf{p}\cdot\mathbf{r})|\psi_{YZ}(\mathbf{r})|^2d^3r,\quad(1)$$

where the T functions are scattering amplitudes for the

* Supported in part by the U. S. Atomic Energy Commission and the Office of Naval Research.
† A portion of this paper was submitted by the author to Carnegie Institute of Technology, Pittsburgh, Pennsylvania, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
1 J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952) Chap. 2, pp. 95–110.
2 G. F. Chew, Phys. Rev. **80**, 196 (1950); S. Fernbach, T. A. Green, and K. M. Watson, *ibid.* **84**, 1084 (1951).
3 J. Ashkin, Suppl. Nuovo Cimento **14**, 238 (1959).
4 A. M. Sachs, H. Winick, and B. A. Wooten, Phys. Rev. **109**, 1733 (1958).
5 K. C. Rogers and L. M. Lederman, Phys. Rev. **105**, 247 (1957).
6 E. Arase, G. Goldhaber, and S. Goldhaber, Phys. Rev. **90**, 160 (1953).
7 E. G. Pewitt, T. H. Fields, G. B. Yodh, J. G. Fetkovich, M. Derrick, preceding paper, Phys. Rev. **131**, 1826 (1963).
8 L. S. Dul'kova, I. B. Sokolova, and M. G. Shafranova, Zh. Eksperim. i Teor. Fiz. **35**, 313 (1958) [translation: Soviet Phys.—JETP **8**, 217 (1959)].
9 M. Goldhaber, Phys. Rev. **101**, 433 (1956); R. F. Christy, in *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957), Sec. 9, p. 1. S. Barshay and M. Schwartz, Phys. Rev. Letters **6**, 421 (1961).
10 E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949); W. Heisenberg, Rev. Mod. Phys. **29**, 269 (1957); Z. Maki, M. Nakagawa, Y. Ohnuki, S. Sakata, Progr. Theoret. Phys. (Kyoto) **23**, 1174 (1960); C. H. Albright, R. Blankenbecler, and M. L. Goldberger, Phys. Rev. **124**, 624 (1961).
11 R. M. Rockmore, Phys. Rev. **105**, 256 (1957).
12 B. H. Bransden and R. G. Moorhouse, Nucl. Phys. **6**, 310 (1958).
13 G. M. Vagradov and I. B. Sokolova, Zh. Eksperim. i Teor. Fiz. **36**, 948 (1959) [translation: Soviet Physics—JETP **9**, 669 (1959)].
14 G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).
15 T. A. Green, Phys. Rev. **90**, 161 (1953); A. Ramakrishnan, V. Devanathan, and K. Venkateson, Nucl. Phys. **29**, 680 (1962).

particles indicated, K is a kinematic factor, and the integral is the form factor of the bound state described by the wave function $\psi_{YZ}(\mathbf{r})$. The variable $\mathbf{p}$ is proportional to the momentum transferred in the collision. In this form-factor approximation the relations between the momenta of W and Y in $T(W,Y)$ are not uniquely determined, neither is the kinematic factor K. This ambiguity arises in the approximation which produces (1) from the correct first-order impulse approximation

$$T(W,X)=\int d^3q\, K(\mathbf{q})T(W,Y;\mathbf{q})\phi_F^*(\mathbf{q})\phi_I(\mathbf{q}) + \text{similar term with } T(W,Z;\mathbf{q}) \quad (2)$$

in which $\mathbf{q}$ is a relative-momentum variable, and $\phi_I(\mathbf{q})$ and $\phi_F(\mathbf{q})$ are Fourier transforms of the wave function $\psi_{YZ}(\mathbf{r})$. The indices I and F refer to initial and final momenta which differ by the momentum transfer $\mathbf{p}$. That is, the form-factor approximation (1) is obtained from the first-order impulse approximation (2) by taking the amplitudes $K(\mathbf{q})$ and $T(W,Y;\mathbf{q})$ out of the integral at some "average values" of $\mathbf{q}$, ignoring all correction terms involving the $\mathbf{q}$ dependence of $K(\mathbf{q})$ and $T(W,Y;\mathbf{q})$: Ambiguities arise because different authors extract different T's and K's. One purpose of this paper is to discuss the "optimal choice" of the average scattering amplitude used in the form-factor approximation and the condition for the validity of such an approximation.

The higher order terms in the impulse series[14] correspond to multiple scattering and potential corrections. Several authors have examined the higher order corrections to the static model,[16] which is not very appropriate for pion-deuteron scattering involving large momentum transfers since in that case large nucleon motions within the deuteron must play an important role. An attempt has been made to include the specific mesonic correction due to meson absorption,[11] but no systematic approach to the problem has been made employing a realistic meson theory throughout. It is desirable to use a meson-theory approach because the multiple scattering of mesons on deuterons cannot be separated from the potential corrections in which mesons are exchanged between the nucleons of the deuteron. In Sec. II a meson theoretic derivation of the impulse series is given in which effects of nucleon motion within the deuteron are specifically retained. The derivation is based on a moving-nucleon generalization of the Heitler-London method[17] for treating two nucleon problems. The derivation is not manifestly Lorentz covariant but the first- and second-order impulse approximations which are obtained are Lorentz covariant. The second-order impulse approximation adds the double-scattering corrections to the first-order approximation. The principal results of Sec. II are summarized in Figs. 1 and 2 which diagrammatically represent the single- and double-scattering contributions to the second-order impulse approximation.

In Sec. III the first-order approximation is reduced to a form-factor expression in an "optimal" way. In order to check the validity of form-factor approximations, the theoretical expressions are computed for charged pions with a laboratory kinetic energy of 142 MeV incident on deuterons, since such calculations can be compared to recent experimental data.[7,18] The form-factor expression is computed in several different ways to assess the importance of: (1) the "optimal" choice of $T(W,Y)$, $T(W,Z)$, and K in Eq. (1) versus other choices,[4,11–13,15,16] (2) the effects of the hard core in nucleon-nucleon interactions on deuteron radial wave functions, (3) the D-state part of the deuteron-state vector, and (4) the S-wave phase shifts in pion-nucleon scattering.

In Sec. IV a form-factor approximation to the double-scattering contribution to the impulse series is found. Arguments are advanced to show that the double-scattering contribution to the impulse series is the only significant part of the multiple-scattering contribution (appropriately defined). The complete form factor approximation is evaluated for two cases: (1) including the D-state wave function in both the single- and double-scattering calculations, and (2) including the D state only in the single-scattering calculation.

In Sec. V the "best" theoretical calculation is compared with the experimental data and found wanting for scattering in the backward hemisphere. An explanation for this discrepancy is proposed which is based on the conditions for the breakdown of a form-factor approximation. Conclusions are drawn concerning the importance of the hard core, the deuteron D state, and the behavior of pion-nucleon scattering amplitudes for unphysical values of their arguments.

II. IMPULSE SERIES

The virtue of the Chew-Low-Wick calculation of pion-nucleon scattering[19] and of Cutkosky's calculation of the two-nucleon potential[20] lies in the use of physical-state vectors to represent nucleons and of physical-meson operators to represent mesons. A physical creation operator is defined as one which creates a physical particle eigenstate of the Hamiltonian when applied once to the physical vacuum state. In the static model the meson operators employed are physical because no antinucleon creation occurs, but a static model is in appropriate in studying pion-deuteron

[16] K. A. Brueckner, Phys. Rev. **89**, 834 (1953); **90**, 715 (1953); S. D. Drell and L. Verlet, *ibid.* **99**, 849 (1956); V. De Alfaro and R. Stroffolini, Nuovo Cimento **11**, 447 (1959); W. Frank and T. Spriggs (to be published).

[17] R. E. Cutkosky, Phys. Rev. **112**, 1027 (1958).

[18] E. G. Pewitt, doctoral dissertation, Carnegie Institute of Technology, 1961 (unpublished).

[19] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956); G. C. Wick, Rev. Mod. Phys. **27**, 339 (1955).

[20] R. E. Cutkosky, Phys. Rev. **116**, 1272 (1959).

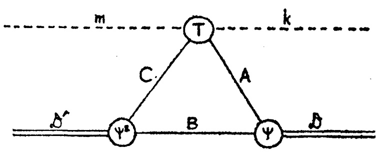

FIG. 1. The single-scattering contribution to the pion-deuteron scattering amplitude.

scattering. The first step in obtaining the impulse series is to transform any given theory (such as the γ_5 theory) into a form closely resembling the static theory.

Do all calculations in one preferred inertial frame using the Schrödinger picture. For heuristic reasons write the Hamiltonian (assumed to exist) as a free part written in terms of bare-meson and bare-nucleon operators and the interaction part multiplied by a switching-on parameter λ (which equals 1 in the real case). Suppose that the physical states are continuous functions of λ. (If necessary, the interaction is provided with appropriate cutoff factors which approach 1 after the calculations are done.) Let bare-meson-destruction operators be denoted by k_0, m_0, q_0, and r_0; bare-nucleon-destruction operators by a_0, b_0, c_0, and d_0.

Find a set of unitary transformations U_λ, continuous in λ, such that $k_\lambda = U_\lambda k_0 U_\lambda^{-1}$ is a physical-meson operator. There are many such U's: The conditions are met if $|k\rangle_\lambda \equiv U_\lambda |k\rangle_0$ is a physical-meson state (for each k) and if $|0\rangle_\lambda \equiv U_\lambda |0\rangle_0$ is the physical-vacuum state. Rewrite the fully-switched-on Hamiltonian in terms of $k_1 = k$ and $a_1 = a$, so that $H = H(a,k;U)$. Impose on U the additional rather vague condition that most of the terms which are neglected in the following calculations are not important because of the form H assumes for the chosen U. In particular, there should be no important terms in $H(a,k)$ which create nucleon-antinucleon pairs. The dominant interaction terms should be those which produce pion emission and absorption by the nucleon core, thus, surrounding the core created by $a^\dagger$ with a physical-meson cloud which provides the interaction between nucleons at energies well below the threshold for real nucleon-antinucleon production. A pion-pion interaction is allowed. It will influence pion-deuteron scattering principally through its effect on the pion-nucleon scattering amplitudes.

Describe the outgoing-wave pion-nucleon scattering state by the vector $|Ak\rangle$, where capital letters for nucleon names (A,B,C,D) indicate that a physical state is being considered. In order to have a compact notation for states involving many mesons, denote a set of mesons (the set may be empty) by a capital such as K, M, Q, or R. Then $|AM\rangle$ will denote a physical state containing a nucleon asymptotically labeled by the parameters A (for momentum, spin, and isotopic spin) and a set of mesons M obeying outgoing-wave boundary conditions. If the set M is empty the physical-nucleon state appears.

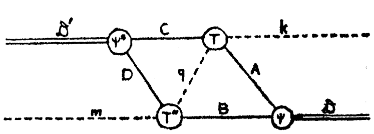

FIG. 2. The double-scattering contribution to the pion-deuteron scattering amplitude.

Describe the physical states involving the deuteron by the vector $|\mathfrak{D}M\rangle$: Thus $|\mathfrak{D}\rangle$ is a deuteron state, and $|\mathfrak{D}k\rangle$ an outgoing pion-deuteron scattering state. Write the Hamiltonian as

$$H = \sum E_k k^\dagger k + V, \tag{3}$$

where $H|k\rangle = E_k|k\rangle$. Define the generalized vertex operators $[MVR]$ recursively by

$$[kQVR] = [k,[QVR]]/n(Q)+1, \quad [QVR]^\dagger = [RVQ], \tag{4}$$

where $n(Q)$ is the number of mesons in set Q. Thus, $[kV] = [k,V]$ and $[Vk] = [V,k^\dagger]$. Note that since the operators a and k commute, the parts of V which consist entirely of core operators do not contribute to (MVR). In the static model only the first vertex operator (kV) is nonzero since V is linear in the k's. One hopes that the best choice of the transformation U will make the $[kV]$ more important than the higher order vertex operators in realistic models also.

The Chew-Low-Wick reduction formulas (5) hold whether Yukawa interactions dominate or not, as one easily checks by multiplying by H:

$$|Ak\rangle = k^\dagger|A\rangle - (H - E_A - E_k - i\epsilon)^{-1}[kV]|A\rangle, \quad |\mathfrak{D}k\rangle = k^\dagger|\mathfrak{D}\rangle - (H - E_{\mathfrak{D}} - E_k - i\epsilon)^{-1}[kV]|\mathfrak{D}\rangle. \tag{5}$$

Furthermore, the scattering amplitudes for pion-nucleon and pion-deuteron scattering follow as before[19];

$$\delta_K(p_A + p_k - p_B - p_m)T(Bm,Ak) = \langle B|[mV]|Ak\rangle, \tag{6a}$$

$$\delta_K(p_{\mathfrak{D}} + p_k - p_{\mathfrak{D}'} - p_m)T(\mathfrak{D}'m,\mathfrak{D}k) = \langle \mathfrak{D}'|[mV]|\mathfrak{D}k\rangle, \tag{6b}$$

where $S_{FI} = \delta_{FI} - 2\pi i\delta(E_F - E_I)\delta_K(p_F - p_I)T(F,I)$ is the S matrix, δ_K is a Kronecker delta function, T is the scattering amplitude if $E_F = E_I$.

In order to reduce (6b) to expressions involving pion-nucleon amplitudes, introduce operators not obeying the usual commutation rules to create physical one-nucleon states (with or without mesonic excitation). Let script capitals $\mathfrak{A}$, $\mathfrak{B}$ denote sets of core and anticore operators. Then

$$|A\rangle = \sum C(\mathfrak{A}M)\mathfrak{A}^\dagger M^\dagger|0\rangle = A^\dagger|0\rangle, \tag{7}$$

where the coefficients $C(\mathfrak{A}M)$ are the wave function of the physical nucleon state in the Fock space of the cores and physical mesons. The coefficients obey the selection rules $p_{\mathfrak{A}} + p_M = p_A$, $q_{\mathfrak{A}} + q_M = q_A$, and $b_{\mathfrak{A}} = 1$, where p_X is the sum of the momenta of the particles in set X, q_X is the sum of the charges, and b_X is the sum of the baryon numbers. Define excited-state operators similarly, separating the operators that create the scattered parts of the physical state:

$$(Ak)^\dagger = k^\dagger A^\dagger + (Ak)_S{}^\dagger, \quad (Ak)^\dagger|0\rangle = |Ak\rangle, \tag{8a}$$

$$(Akm)^\dagger = k^\dagger m^\dagger A^\dagger + k^\dagger (Am)_S{}^\dagger + m^\dagger (Ak)_S{}^\dagger + (km)_S{}^\dagger A^\dagger + (Akm)_S{}^\dagger. \tag{8b}$$

Define Cutkosky's Heitler-London states in the usual manner[17]:

$$|AB\rangle_C = A^\dagger B^\dagger|0\rangle, \tag{9a}$$

$$|ABk\rangle_C = k^\dagger A^\dagger B^\dagger|0\rangle + (Ak)_S{}^\dagger B^\dagger|0\rangle + A^\dagger (Bk)_S{}^\dagger|0\rangle, \tag{9b}$$

and so on, dividing the meson set M of $|ABM\rangle_C$ into three parts in all possible distinguishable ways, assigning one set to scatter from A, another to scatter from B, and a third to interact with neither A nor B (the third part is further subdivided into the various meson-meson scattering possibilities). Let a general Heitler-London state be denoted by the capitals U, W, F, G; e.g., $|W\rangle_C$. The transformation U_λ is so chosen that the states $|W\rangle_C$ span the space of all physical states. For convenience introduce an orthonormal set of Heitler-London states $|W\rangle = \sum \mathfrak{F}_{UW}|U\rangle_C$. Let $\mathfrak{F}$ be a real analytic matrix function of the overlap matrix $\mathcal{G}$ defined by

$$\delta_{UW} + \mathcal{G}_{UW} = \langle U|W\rangle_C. \tag{10}$$

Then $\mathfrak{F} = (1+\mathcal{G})^{-1/2} = 1 - \frac{1}{2}\mathcal{G} + \frac{3}{8}\mathcal{G}^2 + \cdots$, as one easily verifies by computing $\langle U|W\rangle$.

Expand the deuteron state vector and the pion-deuteron scattering-state vector in the orthonormal H$-$L states:

$$|\mathfrak{D}\rangle = \sum |U\rangle\langle U|\mathfrak{D}\rangle, \quad |\mathfrak{D}k\rangle = \sum |U\rangle\langle U|\mathfrak{D}k\rangle. \tag{11}$$

Let $|u\rangle$ denote states in the subset of all the H$-$L states $|W\rangle$ in which no mesonic excitations occur. Then $\langle u|\mathfrak{D}\rangle$ is the usual deuteron momentum wave function ψ (p_A, p_B, spins). Let P be $\sum |u\rangle\langle u|$, the projection operator for the unexcited H$-$L states. Let $P' = 1 - P$. Assume that a good approximation to the deuteron wave function $\langle u|\mathfrak{D}\rangle$ is obtained by solving the reduced Schrödinger equation for a bound state:

$$PHP|\mathfrak{D}_0\rangle = E_0|\mathfrak{D}_0\rangle, \ \mathrm{P}'|\mathfrak{D}_0\rangle = 0, \ \langle u|\mathfrak{D}_0\rangle \approx \langle u|\mathfrak{D}\rangle. \tag{12}$$

Assume further that the wave function $\langle u|\mathfrak{D}_0\rangle$ is known from previous calculations.[1] Obtain an iteration scheme for computing corrections to $|\mathfrak{D}_0\rangle$ by writing $|\mathfrak{D}\rangle = |\mathfrak{D}_0\rangle + |\mathfrak{D}_1\rangle + \cdots$, $E_\mathfrak{D} = E_0 + E_1 + \cdots$, using the idea that the matrix elements of H connecting excited to unexcited states are "smaller" than some of those within the excited and unexcited subspaces. The equations to be solved are

$$PHP|\mathfrak{D}_n\rangle + PHP'|\mathfrak{D}_{n-1}\rangle = \mathrm{P}\sum_{m=0}^{n} E_m|\mathfrak{D}_{n-m}\rangle, \tag{13a}$$

$$P'HP'|\mathfrak{D}_n\rangle + P'HP|\mathfrak{D}_{n-1}\rangle = \mathrm{P}'\sum_{m=0}^{n} E_m|\mathfrak{D}_{n-m}\rangle. \tag{13b}$$

The solution of (13) satisfying the condition $\langle\mathfrak{D}_0|\mathfrak{D}_n\rangle = \delta_{n0}$ is given recursively by

$$E_{2n+1} = 0, \quad E_{2n+2} = \langle\mathfrak{D}_0|H|\mathfrak{D}_{2n+1}\rangle, \quad n = 0, 1, 2, \cdots, \tag{14a}$$

$$|\mathfrak{D}_{2n+1}\rangle = X\left(\sum_{m=0}^{n} E_{2m}|\mathfrak{D}_{2n-2m+1}\rangle - H|\mathfrak{D}_{2n}\rangle\right), \tag{14b}$$

$$|\mathfrak{D}_{2n+2}\rangle = Y\left(\sum_{m=1}^{n+1} E_{2m}|\mathfrak{D}_{2n-2m+2}\rangle - H|\mathfrak{D}_{2n+1}\rangle\right), \tag{14c}$$

where X is the operator inverse to $H - E_0$ in the excited subspace, $X = 0$ in the unexcited subspace; Y is the operator inverse to $H - E_0$ in the part of the unexcited subspace orthogonal to $|\mathfrak{D}_0\rangle$, $Y = 0$ on $|\mathfrak{D}_0\rangle$ and in the excited subspace.

Let the operator L add a meson to the H$-$L states:

$$L|AB\rangle = |ABk\rangle, \quad L|ABm\rangle = |ABkm\rangle, \quad L|ABM\rangle = |ABMk\rangle. \tag{15}$$

Write $|\mathfrak{D}k\rangle = L|\mathfrak{D}\rangle + |\mathcal{S}\rangle$. Obtain the impulse series for $|\mathfrak{D}k\rangle$ by assuming that $|\mathcal{S}\rangle$ is "smaller" than $L|\mathfrak{D}\rangle$ and can be gotten by an iteration scheme reminiscent of the Chew-Low-Wick treatment of the pion-nucleon scattering vector. That is, split the Schrödinger equation

$$H|\mathcal{S}\rangle - E_S|\mathcal{S}\rangle = -HL|\mathfrak{D}\rangle + E_S L|\mathfrak{D}\rangle \equiv -|\mathcal{R}\rangle, \tag{16}$$

where $E_S = E_\mathfrak{D} + E_k$. Write H as a sum of an operator H_0 diagonal in the orthogonal $H-L$ representation with the energies of the separated components of an H$-$L state as its eigenvalues and a remainder H'. Then invert $H - E_S = H_0 - E_S + H'$ with an outgoing wave condition. Expand $(E_S - H_0 - H' + i\epsilon)^{-1}$ in a series assuming H' "small" compared to $E_S - H_0$, obtaining

$$|\mathcal{S}\rangle = \sum_{m=0}^{\infty}[(E_S - H_0 + i\epsilon)^{-1}H']^m (E_S - H_0 + i\epsilon)^{-1}|\mathcal{R}\rangle. \tag{17}$$

Substitute (14), (16), and (17) into (6b). For simplicity in writing formulas keep terms through the "first order of smallness" only; use the fact (to be proven) that $|\mathcal{R}\rangle$ has no zero-order part, thus, obtaining

$$T(\mathfrak{D}'m, \mathfrak{D}k)\delta_K(p_F - p_I) = \langle\mathfrak{D}_0'|(1 - HX)[mV](L - LXH + (E_S - H_0 + i\epsilon)^{-1} \times (H - E_S)L)|\mathfrak{D}_0\rangle + O(2). \tag{18}$$

The fundamental theorem of the Heitler-London approach is that the matrix elements $\langle U|W\rangle_C$, $\langle U|H|W\rangle_C$, and $\langle U|[mV]|W\rangle_C$ can be expanded in a series ordered by the number of mesons exchanged between the nucleons of the two-nucleon H$-$L states; this theorem will be illustrated in subsequent arguments. The matrix element

$$\langle U|H|W\rangle=\sum \mathfrak{F}_{FU}^*\langle F|H|G\rangle_C\mathfrak{F}_{GW}=(\mathfrak{F}^\dagger\mathcal{H}\mathfrak{F})_{UW},$$

where the matrix $\mathcal{H}$ is defined by $\mathcal{H}_{FG}=\langle F|H|G\rangle_C$. By the fundamental theorem and Eq. (10) we can write $H=\sum_{m=0}^\infty H_m$, where $\langle U|H_m|W\rangle=(\sum \mathfrak{F}_n^\dagger\mathcal{H}_p\mathfrak{F}_q)_{UW}$, $n+p+q=m$, and the indices n, p, q, and m denote the number of mesons exchanged in a contribution to a matrix element.

It is a theorem (to be proven) that the series for the operator H' begins with the $m=1$ term; however, the operators L and H_0 are by definition only $m=0$ operators. Expand the operator X in a geometric series similar to that used for $(E_S-H+i\epsilon)^{-1}$. Keep only X_0 since in (18) no second-order terms need be explicitly written; the H associated with X must be really H' since it connects the excited and unexcited spaces only. Use the equation $H_0L|\mathfrak{D}\rangle=\sum L(E_U+E_k)|U\rangle\times\langle U|\mathfrak{D}\rangle$ and the deuteron Schrödinger equation $\sum E_U|U\rangle\langle U|\mathfrak{D}\rangle+H'|\mathfrak{D}\rangle=E_{\mathfrak{D}}|\mathfrak{D}\rangle$ in the equation

$$(H_0+H'-E_{\mathfrak{D}}-E_k)L|\mathfrak{D}\rangle=|\mathfrak{R}\rangle$$

to show that

$$|\mathfrak{R}\rangle=(H'L-LH')|\mathfrak{D}\rangle, \quad (19)$$

thus proving that there is no $m=0$ term in $|\mathfrak{R}\rangle$.

Rewrite (18), decomposing T into zero- and one-meson exchange parts

$$T_0(\mathfrak{D}'m,\mathfrak{D}k)\delta_K(p_F-p_I)=\langle\mathfrak{D}_0'|[mV]_0L|\mathfrak{D}_0\rangle, \quad (20a)$$

$$\begin{aligned}T_1(\mathfrak{D}'m,\mathfrak{D}k)\delta_K(p_F-p_I)\\ =\langle\mathfrak{D}_0'|\{[mV]_1L-H_1(H_0-E_{\mathfrak{D}'})^{-1}(1-P)[mV]_0L\\ +[mV]_0(E_S-H_0+i\epsilon)^{-1}H_1L\\ -[mV]_0(E_S-H_0+i\epsilon)^{-1}LPH_1\}|\mathfrak{D}_0\rangle, \quad (20b)\end{aligned}$$

using

$$L(H_0-E_{\mathfrak{D}})^{-1}(1-P)+(E_S-H_0+i\epsilon)^{-1}L(1-P)=0.$$

The no-meson-exchange term (20a) yields the simple impulse approximation: The incoming meson strikes one nucleon, then goes away as the scattered meson. The one-meson-exchange terms (20b) contain deuteron distortion effects as well as double scattering contributions to the impulse series. The deuteron-distortion effects are all produced by the incoming meson. None of them are hidden attempts of the meson interaction to form the deuteron wave function as one easily checks by comparing the perturbation expansion of (20b) with an ordinary perturbation expansion containing the characteristic ladder denominators.

To illustrate the fundamental theorem consider the overlap-matrix element $\langle U|W\rangle_C$ for the case that U and W describe unexcited states: $\langle 0|BAC^\dagger D^\dagger|0\rangle$. Push the creation operator C^+ to the left using the commutation rules for the core operators contained in C^+, A, and B. There will result terms involving commutators of all the core operators in $C^\dagger$ with all those in either A ($C^\dagger$ is absorbed by A) or in B ($C^\dagger$ is absorbed by B). There will also result terms involving commutators of some $C^\dagger$ cores with A cores and some with B cores; call such terms core-exchange terms. Since core-exchange terms do not appear in static theory consider them to be small short-range corrections which need not be written out explicitly. Denote the destiny of the core operators of $C^\dagger$ by a subscript,

$$\begin{aligned}\langle 0|BAC^\dagger D^\dagger|0\rangle=\langle 0|BAC_A{}^\dagger D^\dagger|0\rangle+\langle 0|BAC_B{}^\dagger D^\dagger|0\rangle\\ +\text{core exchange terms (c.e.)}. \quad (21)\end{aligned}$$

Use the anticommutation relations for A and B and also for B and $C_A{}^\dagger$ to obtain

$$\begin{aligned}\langle 0|BAC^\dagger D^\dagger|0\rangle\\ =-\langle 0|A\{C_A{}^\dagger,B\}D^\dagger|0\rangle+\langle 0|AC_A{}^\dagger BD^\dagger|0\rangle\\ +|0|B\{C_B{}^\dagger,A\}D^\dagger|0\rangle-\langle 0|BC_B{}^\dagger AD^\dagger|0\rangle+\text{c.e.} \quad (22)\end{aligned}$$

A typical term of the anticommutator $\{C_A{}^\dagger,B\}$ is $\{\mathfrak{A}^\dagger M^\dagger,\mathfrak{B}Q\}c(\mathfrak{A}M)d^*(\mathfrak{B}Q)$. The sets $\mathfrak{A}$, $\mathfrak{B}$ must anticommute because they contain odd numbers of core operators and are not "allowed" to annihilate one another. Therefore, the term becomes $c(\mathfrak{A}M)d^*(\mathfrak{B}Q)\times\mathfrak{A}^\dagger\mathfrak{B}[M^\dagger,Q]$. To evaluate $[M^\dagger,Q]$ use the identity

$$[M^\dagger,Q]=-\sum_R{}'[[r,M^\dagger]][[Q,r^\dagger]], \quad (23)$$

where $[[r,M^\dagger]]$ is a compact notation for the nested commutator $[r_1,[r_2,\cdots[r_n,M^\dagger]\cdots]]$, $[[M,r^\dagger]]$ is its Hermitian conjugate, the sum runs over all sets of mesons R (except the empty set) in all permutations, and each term is divided by $n!$. Rewrite (22) as

$$\begin{aligned}\langle 0|BAC^\dagger D^\dagger|0\rangle=\sum_R\langle 0|A[[r,C_A{}^\dagger]]\,[[B,r^\dagger]]D^\dagger|0\rangle\\ -\sum_R\langle 0|B[[r,C_B{}^\dagger]]\,[[A,r^\dagger]]D^\dagger|0\rangle+\text{c.e.}, \quad (24)\end{aligned}$$

where the absence of the prime on the summation on R means that the empty set is allowed. Insert a unit operator in the center of these expressions, keeping only the part $\sum_Q Q^\dagger|0\rangle\langle 0|Q$ involving meson operators, since core-anticore terms will imply short-range core-exchange corrections. Push the meson operators Q of the complete set through the commutators such as $[[r,C_A{}^\dagger]]$ obtaining $\sum_{RQ}\langle A|Q^\dagger R|C\rangle\langle B|R^\dagger Q|D\rangle$ for the first term of (24); (24) becomes

$$\begin{aligned}\langle 0|BAC^\dagger D^\dagger|0\rangle=\sum_{RQ}\langle A|Q^\dagger R|C\rangle\langle B|R^\dagger Q|D\rangle\\ -\sum_{RQ}\langle B|Q^\dagger R|C\rangle\langle A|R^\dagger Q|D\rangle+\text{c.e.} \quad (25)\end{aligned}$$

Use the identity

$$rA^\dagger|0\rangle=-(H+E_r-E_A)^{-1}[rV]A^\dagger|0\rangle \quad (26)$$

to remove the meson operators from the terms of (25) involving only one meson operator in the one-nucleon matrix elements. These terms then produce one-meson exchange contributions to the overlap matrix element such as

$$\begin{aligned}\langle A|[rV]|C\rangle\langle B|[Vr]|D\rangle\\ \times(E_A-E_C+E_r)^{-1}(E_B-E_D-E_r)^{-1}.\end{aligned}$$

Use the identity

$$r(H-E)^{-1}=(H+E_r-E)^{-1}r-(H+E_r-E)^{-1}\times[rV](H-E)^{-1} \quad (27)$$

to reduce terms such as $\langle A|rq|C\rangle$, so that no meson operators remain when the inevitable sum over intermediate states is introduced. As a consequence of this precaution which must always be taken in order to increase the rate of convergence of the series, no singular $\delta(r,k)$ factors will arise from the action of r on an intermediate state containing a scattering meson k.

To see what happens when the matrix element $\langle U|W\rangle_C$ involves excited $H-L$ states, consider $\langle mBA|CDk\rangle_C$. The only singular parts $\delta(m,k)$ that arise come from the term $\langle 0|mBAC^\dagger D^\dagger k|0\rangle$, one of the nine terms entailed by (9b), since the scattering part terms have the meson operator removed by (5). The precaution embodied in (27) assures that no singular factor $\delta(k,r)$ or $\delta(m,r)$ need ever occur in matrix elements. One can identify the terms that do produce δ factors in a unique manner, thus obtaining a decomposition of a matrix element into a sum of products of nonsingular factors and δ factors. The nonsingular factors are called irreducible;[17] a subscript I on a matrix element means to take the irreducible part, which can be obtained by requiring mesons such as m and k never to annihilate one another. The decomposition theorem states that

$$\begin{aligned}\langle mBA|CDk\rangle_C &= \delta(m,k)\langle 0|BAC^\dagger D^\dagger|0\rangle+\langle 0|B(Am)(Ck)^\dagger D^\dagger|0\rangle_I \\ &+\langle 0|(Bm)A(Ck)^\dagger D^\dagger|0\rangle_I+\langle 0|B(Am)C^\dagger(Dk)^\dagger|0\rangle_I \\ &+\langle 0|(Bm)AC^\dagger(Dk)^\dagger|0\rangle_I, \quad (28)\end{aligned}$$

with similar expressions for more meson-laden matrix elements.

To evaluate $\langle U|H|W\rangle_C$ divide the operator V into a meson-meson interaction part V' and a part containing core operators V''. Push the operator V'' to the right; labeling V'' by the operator upon which it dies, obtain

$$\begin{aligned}HC^\dagger D^\dagger|0\rangle &= (E_C+E_D)C^\dagger D^\dagger|0\rangle+[V_D''+V', C^\dagger]D^\dagger|0\rangle \\ &-[V_C''+V', D^\dagger]C^\dagger|0\rangle+V'C^\dagger D^\dagger|0\rangle+\text{c.e.} \quad (29)\end{aligned}$$

Use (23) to obtain

$$[V_D''+V', C^\dagger]=\textstyle\sum_R'[[r,C^\dagger]]\,[[V_D''+V', r^\dagger]]. \quad (30)$$

Push the operator $[[r,C^\dagger]]$ to the left in

$$\langle 0|BAHC^\dagger D^\dagger|0\rangle,$$

labeling it by the operator upon which it dies, etc. The resulting expression contains a term $(E_C+E_D)\times\langle 0|BAC^\dagger D^\dagger|0\rangle$ and a sum of four interaction terms such as $H_{AC,BVD}$, where

$$H_{AC,BVD}=\textstyle\sum_R'\sum_Q\sum_M\langle A|M^\dagger QR|C\rangle\times\langle B|Q^\dagger M[VR]|D\rangle n(R)!. \quad (31)$$

Note that in the general matrix element the factor $(\delta_{UW}+\mathcal{G}_{UW})E_W$ will become δ_{FG} when the transition to the orthonormal $H-L$ matrix element $\langle F|H|G\rangle$ is made via the $\mathfrak{F}$ transformation.

Further reduce the result (31) by pushing the operators M through the generalized vertex operator $[VR]$ producing vertex factors such as $[M_1VR]M_2$, where $M_1+M_2=M$.[17] Symmetrize the derivation by pushing all V's to the left, finding that $\langle 0|BAHC^\dagger D^\dagger|0\rangle$ consists of the average energy times $\langle 0|BAC^\dagger D^\dagger|0\rangle$ plus half the sum of eight antisymmetrical terms like $H_{AC,BVD}$ (all distinct sensible permutations of the indices appear) plus a meson-meson term $\langle 0|BAV'C^\dagger D^\dagger|0\rangle$ plus core-exchange terms.

The decomposition theorem for matrix elements of H is illustrated by

$$\langle mBA|H|CDk\rangle_C=\delta(k,m)\langle BA|(H+E_k)|CD\rangle_C+\langle mBA|H|CDk\rangle_{CI}. \quad (32)$$

Derive a meson-exchange series for matrix elements $\langle U|[mV]|W\rangle_C$ in a manner similar to that used for matrix elements of H, observing that the following differences occur: There are no kinetic-energy terms such as $E_W\langle U|W\rangle_C$, and the terms such as (31) which appear have $[mV]$ replacing V, and the summation over R is extended to include the empty set.

Most terms in the impulse series for T [begun by (20a), (20b)] involve products of the matrices $\mathcal{G}_n$, $\mathcal{H}_n$, $[mV]_{Cn}$ (n is the number of mesons exchanged) evaluated between deuteron wave functions $\langle u|\mathfrak{D}_0\rangle$, although a few terms contain only one matrix. To evaluate these matrix products perform a sum over all intermediate states, noting that by the definition of the matrices the sum is over all distinct nonorthogonal H$-$L states. To simplify later integrations sum over all momentum variables for the particles in the H$-$L states independently, taking care to antisymmetrize nucleon factors appropriately and to divide by the correct overcounting factors. Note that in the contributions from intermediate states containing excited mesons a meson may be attached to one nucleon in the final state of one matrix and to the other nucleon in the initial state of the next matrix to the left: In a diagrammatic analysis of the impulse series it appears that a meson is being exchanged between the nucleons. Such an exchanged meson (called an excited meson) has an effect different from that of an ordinary exchanged meson such as occurs in a diagrammatic analysis of $\mathcal{G}_n$. The excited meson is in a scattered wave with respect to a nucleon, whereas an exchanged meson is in the cloud of a physical nucleon (the unscattered mesons in excited H$-$L states are not associated with either nucleon). The impulse series is a double series in the numbers of mesons exchanged and the number of mesons excited.

The contribution to the impulse series from the zero meson exchange (20a) is

$$T_0(\mathfrak{D}'m,\mathfrak{D}k)=2(2\pi)^{-3}\Omega\int\psi^*(CB,\mathfrak{D}')T(Cm,Ak)\times\psi(AB,\mathfrak{D})d^3A\,, \quad (33)$$

where the deuteron wave function

$$\langle AB|\mathfrak{D}_0\rangle=\sqrt{2}\psi(AB,\mathfrak{D})\delta_K(p_A+p_B-p_{\mathfrak{D}});$$

the four-vertex function

$$\langle C|[mV]|Ak\rangle=\delta_K(p_A+p_k-p_C-p_m)T(Cm,Ak)$$

is the pion-nucleon scattering amplitude (6a) when the momenta are such as to conserve energy, Ω is the quantization volume, and the momenta are constrained to conserve momentum in each factor in (33).

The contributions from the one-meson-exchange terms (20b) are either of the potential correction form or the double scattering form. A typical one of the four simplest potential contributions is

$$T_{1p1}=2(2\pi)^{-6}\Omega^2\int\int\psi^*(DC,\mathfrak{D}')v^*(BC,D)T(Cq,A'k)\times v(A'm,A)\psi(AB,\mathfrak{D})h_{1p1}(E_i)d^3Ad^3q\,, \quad (34)$$

where $\langle B|[qV]|D\rangle=v(Bq,D)\delta_K(p_D+p_q-p_B)$ and $h_{1p1}(E_i)$ is a function of the energies of all the particles involved in (34) which reduces to $(E_qE_k)^{-1}$ in the static unbound limit (the limit obtained by allowing the masses of nucleons to approach infinity and by neglecting the binding energy of the deuteron). The three other simple potential contributions differ from T_{1p1} in the energy functions $h(E_i)$ and in the order in which the mesons k, q, and m appear in the vertex functions. In two of them the four-vertex function to be evaluated is of the form $\langle C|[Vq]|A'k\rangle$ instead of $\langle C|[qV]|A'k\rangle$. In a diagrammatic analysis these four terms describe the four processes in which the meson k scatters off one nucleon to be absorbed on the other as meson q, while the final-state meson m is emitted by the first nucleon. The meson m may be emitted before or after the scattering of k, which may scatter into a meson q moving either forward or backward in time. These four simple terms come from the "no-mesons excited" part and the "one-meson excited but not scattering" part of the meson-excitation series for the one-meson exchange contribution to the impulse series; the terms in quotations refer to mesons in the intermediate states of matrix products in (20b). The "one-meson excited and scattering" part and multimeson-excitation parts of the excitation series provide terms containing the amplitude for meson production in meson-nucleon scattering; omit such terms on the grounds that the meson production amplitude will be small for energies of the incident pion below 300 MeV.

Of the eight simplest double-scattering contributions, the dominant one has the form

$$T_{1S1}=2(2\pi)^{-9}\Omega^3\int\int\int d^3Ad^3qd^3q'\,\psi^*(CD,\mathfrak{D}')T(Dm,B'q')\times T'(B'q',Bq)T(Cq,Ak)\psi(AB,\mathfrak{D})h_{|S|}(E_i)\,, \quad (35)$$

where

$$\langle B'q'|[Vq]|B\rangle=T'(B'q',B_q)\delta_K(p_B'+p_q'-p_B-p_q).$$

The four-vertex function T' is not the pion-nucleon scattering amplitude because the state $|B'q'\rangle$ contains outgoing waves, not incoming ones. In a diagrammatic analysis (35) represents a meson k scattering off one nucleon to the other which enters an excited state eventually emitting the final meson m. There are three other terms in which either the second nucleon does not enter the excited state or the final meson is emitted before the exchanged meson is absorbed or both of these things happen. There are four more terms which differ from the first four only in that the exchanged meson propagates backward in time with the corresponding alteration of the energy functions $h(E_i)$.

In the static-unbound limit the energy functions for the first four terms are products of two functions: One is a propagator for the exchanged meson q and the other is the energy denominator appearing in the Low-equation expansion of the scattering amplitude $\langle S|[Vq]|B\rangle=\delta_K(p_0+p_m-p_B-p_q)T''(Dm,Bq)$, where $|S\rangle$ is the scattering state containing nucleon D and meson m with ingoing-wave boundary conditions. The Low-equation expansion, not in the static unbound limit, is

$$\begin{aligned}\langle S|[Vq]|B\rangle &= (E_D+E_m-E_B')^{-1}\langle D|[mV]|B'\rangle\langle B'|[Vq]|B\rangle \\ &-(E_B''+E_m-E_B)^{-1}\langle D|[Vq]|B''\rangle\langle B''|[mV]|B\rangle \\ &+\textstyle\sum_{B'q'}(E_D+E_m+i\epsilon-E_B'-E_q')^{-1}\langle D|[mV]|B'q'\rangle \\ &\times\langle B'q'|[Vq]|B\rangle-\textstyle\sum_{B'q'}(E_B'+E_q'+E_m-E_B)^{-1} \\ &\times\langle D|[Vq]|B'q'\rangle\langle B'q'|[mV]|B\rangle+\cdots. \quad (36)\end{aligned}$$

By using the Low equation make the first four terms yield the expression

$$T_{DS}=2(2\pi)^{-6}\Omega^2\int\int d^3Ad^3q\,\psi^*(CD,\mathfrak{D}')T''(Dm,Bq)\times T(Cq,Ak)\psi(AB,\mathfrak{D})h_{DS}(E_i) \quad (37)$$

plus four terms which vanish in the static unbound limit. The other four terms of the simplest double-scattering contribution have large energy denominators because of the "backwardness" of the exchanged meson, so their contribution is probably small compared to that of (37).

The propagator $h_{DS}(E_i)$ splits into an energy delta function and a part which yields a principal value integral. Because the T amplitudes will contribute mainly near their resonance value the principal-value integral will be small and out of phase with the energy-shell contribution and the single-scattering contribution. In the following calculations neglect the principal-value part of (37), the other double-scattering contributions, the potential contributions, and all multimeson exchange and excitation contributions. The resulting approximation to the impulse series is represented by the diagrams of Fig. 1 and Fig. 2.

III. SINGLE-SCATTERING APPROXIMATIONS

The single-scattering contribution T_0 to the pion-deuteron scattering amplitude [(33), Fig. 1] is given in the pion-deuteron barycentric frame by

$$T_0=2(2\pi)^{-3}\Omega\int d^3p_A\,\psi^*(p_C\sigma_C\tau_C,p_B\sigma_B\tau_B;S_F)\times T(p_C\sigma_C\tau_C,m-;p_A\sigma_A\tau_A,k-)\times\psi(p_A\sigma_A\tau_A,p_B\sigma_B\tau_B;S_I),\quad(38)$$

where the spin and isospin variables implicit in (33) are explicitly indicated (the summation convention applies to them), S_F and S_I are spin variables for the initial and final deuteron, and the minus sign indicates the meson charge. The physical significance of (38) is easily inferred from Fig. 1: The reaction $\mathfrak{D}\rightarrow A+B$ proceeds virtually with amplitude $\psi(AB,\mathfrak{D})$, followed by the recombination $B+C\rightarrow\mathfrak{D}'$ with amplitude $\psi^*(BC,\mathfrak{D}')$; the amplitudes for all possible reaction chains are added to give the $\pi-D$ scattering amplitude. Because the π^+-D system is charge symmetric to the π^--D system, (38) also holds for π^+ scattering if Coulomb effects are ignored (which is justifiable for scattering angles greater than 30° for incident-pion energies above 100 MeV).[11]

The deuteron–momentum–space wave function $\psi(AB,\mathfrak{D})$ is a Fourier transform of the ordinary deuteron wave function if the correct Lorentz transformation to the deuteron rest frame is approximated by a Galilean transformation,

$$\psi(AB,\mathfrak{D})=\Omega^{-1/2}I_0(\tau_A,\tau_B)\times\int d^3x\exp[-\tfrac{1}{2}i(\mathbf{p}_A-\mathbf{p}_B)\cdot\mathbf{x}]\phi(\mathbf{x}\sigma_A\sigma_B,S\mathfrak{D}),\quad(39)$$

where $\mathbf{x}$ is the relative coordinate $\mathbf{x}_A-\mathbf{x}_B$, $\mathbf{p}_A+\mathbf{p}_B=\mathbf{p}_\mathfrak{D}=-\mathbf{p}_k$, and I_0 is the antisymmetric isospin function. The wave function $\phi(x\sigma_A\sigma_B,s)$ can be written as a linear combination of the S-state radial function $u(r)$ and the D-state radial function $w(r)$:

$$\phi(\mathbf{x}\sigma_A\sigma_B,S)=\textstyle\sum_n r^{-1}[u(r)Y_{00}(\mathbf{x})\delta_{Sn}+w(r)Y_{2,S-n}(\mathbf{x})a_{Sn}]\chi_n(\sigma_A\sigma_B)=\sum_n\phi_n\chi_n,\quad(40)$$

where the $Y_{lm}(\mathbf{x})$ are the spherical harmonics, the χ_n are symmetric spin functions, and the a_{Sn} are vector-coupling coefficients.[1]

On the energy shell the pion-nucleon scattering amplitude $T(Cm,Ak)$ is related to the pion-nucleon barycentric amplitude $T_B(C'm',A'k')$ by[21]

$$[E_AE_CE_kE_m]^{1/2}T(Cm,Ak)=E_A'E_k'T_B(C'm',A'k'),\quad(41)$$

where the primed variables are the appropriate Lorentz transforms of the unprimed ones. Since the velocity associated with the transformation is less than $0.1c$ for the momenta involved in the present calculation a semi-Galilean transformation may be used instead of the Lorentz transformation without incurring errors larger than a few percent. Under semi-Galilean transformation the nucleon spin variables are unaltered, so that (41) need not be limited to helicity amplitudes[22] but may be used for spin amplitudes defined with respect to a fixed frame, the pion-deuteron barycentric frame. The semi-Galilean transformations are defined by

$$\mathbf{k}'=\mathbf{k}-\mathbf{v}E_k,\quad\mathbf{v}=\tfrac{1}{4}(\mathbf{m}+\mathbf{k})(E_A+E_k)^{-1}.\quad(42)$$

The scattering amplitude T_B may be adequately represented in the energy region below 200 MeV by an expansion in S and P partial waves:

$$T_B=-2\pi\Omega^{-1}v_Rk'^{-2}\textstyle\sum_n\sin\delta_n\exp(i\delta_n)\times h_n(\sigma_C\tau_C\sigma_A\tau_A)=\sum t_nh_n,\quad(43)$$

where v_k is the relative velocity of the nucleon and pion in the pion-nucleon barycentric frame, δ_n is the phase shift in the eigenstate labeled by n, and the h_n are projection functions associated with the eigenstates n. As an example, the $(\frac{3}{2},\frac{3}{2})$ function in matrix notation follows:

$$h_{33}=(\tfrac{2}{3}+\tfrac{1}{3}\tau_3)[3\mathbf{k}'\cdot\mathbf{m}'-(\boldsymbol{\sigma}\cdot\mathbf{m}')(\boldsymbol{\sigma}\cdot\mathbf{k}')]k'^{-2}.\quad(44)$$

As a consequence of the factorization of the spin dependences of ψ and T [(40), (43)] after summing over all spins the single-scattering amplitude T_0 assumes the form

$$T_0=2(2\pi)^{-3}\int d^3p_Bd^3xd^3y\textstyle\sum b_{nrq}(\mathbf{p}_B)\times\exp[-\tfrac{1}{2}i(\mathbf{k}\cdot\mathbf{y}-\mathbf{m}\cdot\mathbf{x})]\exp[i\mathbf{p}_B(\mathbf{x}-\mathbf{y})]\times\phi_n^*(\mathbf{x})t_r(\mathbf{p}_B)\phi_q(\mathbf{y}),\quad(45)$$

where the spin-summation coefficients b_{nrq} depend on $\mathbf{p}_B$ slightly since the transformation factor of (41) and the pion-nucleon barycentric scattering angles depend on $\mathbf{p}_B$. In the static approximation t_r is independent of $\mathbf{p}_B$ as are the b_{nrq} so that the integrations over $\mathbf{p}_B$ and $\mathbf{y}$ may be easily done, yielding the form-factor approximation

$$T_{00}=2\textstyle\sum b_{nrq}t_r\int d^3x\,\phi_n^*(\mathbf{x})\phi_q(\mathbf{x})\times\exp[-\tfrac{1}{2}i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}].\quad(46)$$

[21] C. Møller, Kgl. Danske Videnskab. Selskab. Mat. Fys. Medd. 23, 1 (1945).

[22] M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

In order to obtain a form-factor formula without taking the drastic step of assuming a static approximation, expand $b_{nrq}t_r$ in a Taylor series in $\mathbf{p}_B$ about some as yet undetermined $\mathbf{p}_0$. If the other $\mathbf{p}_B$-dependent factor multiplying $b_{nrq}t_r$ in (45) is sharply peaked near $\mathbf{p}_0$, and if $b_{nrq}t_r$ is slowly varying near $\mathbf{p}_0$, it is a reasonable approximation to neglect all but the first term in the power series, thus obtaining the form-factor formula (46). In order to choose the best form-factor approximation from the three-parameter family obtained by varying $\mathbf{p}_0$, choose $\mathbf{p}_0$ so that the second term in the Taylor series vanishes identically when the $\mathbf{p}_B$ integration is performed. The second term is of the form

$$T_0'=2\nabla_p[\sum b_{nrq}(\mathbf{p})t_r(\mathbf{p})]_0\cdot\int d^3pd^3yd^3x(\mathbf{p}-\mathbf{p}_0)\times\exp[-\tfrac{1}{2}i(\mathbf{k}\cdot\mathbf{y}-\mathbf{m}\cdot\mathbf{x})]\exp[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})]\times\phi_n^*(\mathbf{x})\phi_q(\mathbf{y}). \quad (47)$$

Requiring that the contribution to (47) from a pure S-wave deuteron vanish identically, and making the substitutions $\mathbf{y}=\mathbf{R}+\frac{1}{2}\mathbf{r}$, $\mathbf{x}=\mathbf{R}-\frac{1}{2}\mathbf{r}$, one finds easily that $\mathbf{p}_0=-\frac{1}{4}(\mathbf{m}+\mathbf{k})$. This choice of $\mathbf{p}_0$ has the advantage that the resulting pion-nucleon scattering amplitude is on the energy shell [the energy shell condition is that $\mathbf{p}_B\cdot(\mathbf{m}-\mathbf{k})=0$] so that (41) correctly describes the transformation properties of the preferred T. Note that the "optimal" choice $\mathbf{p}_0=-\frac{1}{4}(\mathbf{m}+\mathbf{k})$ depends on the angle of scattering so the phase shifts appearing in (46) will also depend on the scattering angle. Previous calculations are equivalent to making the "forward" choice $\mathbf{p}_0=-\frac{1}{2}\mathbf{k}$, which is independent of scattering angle and which puts the T amplitude off the energy shell except for forward scattering. Pion-deuteron cross sections computed with the optimal choice and with the forward choice can be found in Table I; they differ by 30% for backward scattering.

In order to compute the cross sections the following prescriptions are used: The pion-nucleon phase shifts are taken from the effective-range formulas[3]

$$\cot\delta_{33}=m_0^2E^*(\tfrac{4}{3}f^2k'^3)^{-1}(1-E^*/E_0),\quad \delta_{31}=\delta_{13}=\delta_{11}=0$$
$$\delta_3=0.11k'/m_0,\quad \delta_1=-0.17k'/m_0, \quad (48)$$

where $E^*=[k'^2+m_0^2]^{1/2}+k'^2/2M$, $E_0^*=308$ MeV, $m_0=140$ MeV, $f^2=0.08$, $M=939$ MeV. The deuteron-radial wave functions are written as $u(r)=v(r)\cos\gamma(r)$, $w(r)=v(r)\sin\gamma(r)$, where the function $\gamma(r)=0$ for calculations ignoring the D state, and for calculations including the D state $\gamma(r)$ is obtained numerically[23] from approximate solutions of the Schrödinger equation containing a meson-theoretic tensor potential.[24,20] The function $v(r)$ is chosen to have a modified Hulthén shape

$$v(r)=N\{\exp[-\alpha(r-r_0)]-\exp[-\beta(r-r_0)]\},\quad r>r_0$$
$$=0,\quad r<r_0 \quad (49)$$

which fits the binding energy ($\alpha=45.7$ MeV), the existence of a repulsive core with a radius of $\frac{1}{3}$ the pion reduced Compton wavelength ($r_0=(420\text{ MeV})^{-1}$ corresponding to 0.47×10^{-13} cm), and the triplet effective range $r_E=1.70\times10^{-13}$ cm ($\beta=494$ MeV, $N^2=121$ MeV). If the core radius is set equal to zero, $\beta_0=282$ MeV, $N_0^2=151$ MeV.

The pion-deuteron barycentric differential cross section, summed over all final-deuteron spin states, and averaged over all initial spin states, is given by

$$d\sigma/d\Omega=\tfrac{1}{3}\Omega^2k^2v_{\text{c.m.}}^{-2}(2\pi)^{-2}\sum_{IF}|T(S_F,S_I)|^2, \quad (50)$$

where $v_{\text{c.m.}}$ is the relative velocity $k/E_k+k/E_{\mathfrak{D}}$. The computation of the spin-summation coefficients of (45) may be effected without any difficulty, and the angular integrations performed in (46) by expanding the product $\phi_n^*(\mathbf{x})\phi_q(\mathbf{x})$ and the exponential in spherical harmonics. Inserting the form-factor approximation (46) to the single-scattering amplitude (38) into (50) yields the following expression for the cross section, where use has *not* been made of the vanishing of the small P-wave pion-nucleon phase shifts:

$$d\sigma/d\Omega=(4k^2v_R^2E_k'^2v_{\text{c.m.}}^{-2}k'^{-4}E_k^{-2})\times[S_0^2(b)|Z_0(\theta)|^2+\tfrac{2}{3}S_1^2(b)|Z_1(\theta)|^2], \quad (51)$$

where

$$S_1(b)=\{(j_0A)^2+4[(j_2B)-\tfrac{1}{4}\sqrt{2}(j_2C)]^2\}^{1/2},$$
$$S_1(b)=(j_0A)-\tfrac{3}{2}(j_0C)+\tfrac{1}{2}\sqrt{2}(j_2B)+\tfrac{1}{2}(j_2C),$$
$$(j_0A)=\int_0^\infty j_0(br)[u^2(r)+w^2(r)]dr,$$
$$(j_2B)=\int_0^\infty j_2(br)u(r)w(r)dr,$$
$$(j_0C)=\int_0^\infty j_0(br)w^2(r)dr,$$
$$(j_2C)=\int_0^\infty j_2(br)w^2(r)dr,$$
$$Z_0=\cos\theta(\tfrac{4}{3}\eta_{33}+\tfrac{2}{3}\eta_{31}+\tfrac{2}{3}\eta_{13}+\tfrac{1}{3}\eta_{11})+\tfrac{2}{3}\eta_3+\tfrac{1}{3}\eta_1,$$
$$Z_1=\sin\theta(\tfrac{2}{3}\eta_{31}-\tfrac{2}{3}\eta_{33}+\tfrac{1}{3}\eta_{11}-\tfrac{1}{3}\eta_{13}),$$
$$\eta_{mn}=\sin\delta_{mn}\exp(i\delta_{mn}),\quad b=\tfrac{1}{2}|\mathbf{m}-\mathbf{k}|,$$
$$\cos\theta=(\mathbf{k}'\cdot\mathbf{m}')/k'^2,$$

and $j_l(x)$ is a spherical Bessel function of order l. The expression $S_1(b)$ can be considered a spin-flip

[23] H. D. Young and R. E. Cutkosky, Phys. Rev. **117**, 595 (1960).

[24] J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, Suppl. Progr. Theoret. Phys. (Kyoto) **3**, 32 (1956).

TABLE I. Pion-deuteron barycentric differential cross section versus scattering angle for 142-MeV laboratory pions, in mb/sr.

a	0°	30°	60°	90°	120°	150°	180°
σ_{SM}	20.6	11.4	2.73	0.62	0.82	1.26	1.44
σ_{FC}	40.2	22.2	5.33	1.20	1.61	2.47	2.82
σ_{OC}	40.2	23.2	6.16	1.57	2.31	3.71	4.30
σ_{S}	40.6	22.2	5.17	1.66	2.64	3.73	4.13
σ_{SC}	40.6	21.5	4.95	1.32	2.25	3.35	3.61
σ_{SCD}	40.6	21.2	4.75	1.08	2.24	3.57	3.93
σ_{DSS}	38.5	19.6	3.85	0.78	1.63	2.42	2.47
σ_{best}	38.3	19.6	3.86	0.88	1.60	2.45	2.50
σ_{expt}		18.0	2.7	1.20	0.90	0.90	1.30
uncertainty		3.0	0.5	0.14	0.10	0.13	0.21

[a] SM, static model; FC, forward choice; OC, optimal choice; S, S-phases added; SC, core also added; SCD, *D* state also added; DSS, double-scattering *S* state only added; best, double scattering *D* state also added; expt, experimental results of Pewitt *et al.* (Refs. 7, 18); uncertainty, experimental uncertainty.

form factor of the deuteron, $S_0(b)$ a "direct" one. In making static-model calculations various authors use various approximations to (51) which will be discussed briefly in Sec. V; one of the most inconsistent of these is entered in Table I because of its reputed agreement with experiment.[12] The effects of successively adding the pion-nucleon S-wave phase shifts, the repulsive core, and the deuteron D state to an "optimal choice" evaluation of (51) are also shown in cross sections entered in Table I. The small-angle cross sections are for comparison of the different calculations only since the Coulomb effects have been neglected. For simplicity, the static model and forward-choice cross sections have been evaluated with no S-wave phase shifts, no repulsive core, and no deuteron D state.

IV. DOUBLE-SCATTERING APPROXIMATIONS

The double-scattering contribution T_{DS} [(37), Fig. 2] is given by

$$T_{DS}=-2\pi i(2\pi)^{-6}\Omega^2\int d^3p_A d^3q\,\psi^*(CD,\mathfrak{D}')T''(Dm,Bq)$$
$$\times\delta(E_q+E_C-E_k-E_A)T(Cq,Ak)\psi(AB,\mathfrak{D}),\quad(52)$$

where a summation over the spin and isospin indices for the nucleons and the isospin index for the intermediate pion is implied. The physical significance of each factor in the formula is easily inferred from Fig. 2; the Dirac delta function puts the intermediate pion on the energy shell for the first scattering. Since the first scattering is on the energy shell, the second should not be far off it, making it a reasonable approximation to replace T'' by T. Employing Galilean and semi-Galilean transformations as in Sec. III, one may write each of the amplitudes in (52) as a sum of products of spin factors and either wave functions or scattering functions. Making the substitutions $\mathbf{x}=\mathbf{R}-\frac{1}{2}\mathbf{r}$, $\mathbf{y}=\mathbf{R}+\frac{1}{2}\mathbf{r}$ leads to an expression analogous to (45),

$$T_{DS}=-2\pi i(2\pi)^{-6}\Omega\int d^3p_A d^3q d^3R d^3r\sum b_{nruj}(\mathbf{p}_A,\mathbf{q})$$
$$\times\exp[\tfrac{1}{2}i(\mathbf{k}+\mathbf{m}-2\mathbf{q})\cdot\mathbf{R}]$$
$$\times\exp[i(-\tfrac{3}{4}\mathbf{k}-\tfrac{1}{4}\mathbf{m}+\tfrac{1}{2}\mathbf{q}-\mathbf{p}_A)\cdot\mathbf{r}]\phi_n^*(\mathbf{R}-\tfrac{1}{2}\mathbf{r})$$
$$\times t_r(\mathbf{p}_A,\mathbf{q})\delta(E_q+E_C-E_k-E_A)$$
$$\times t_u(\mathbf{p}_A,\mathbf{q})\phi_j(\mathbf{R}+\tfrac{1}{2}\mathbf{r}),\quad(53)$$

where the b_{nruj} are spin-summation coefficients which are easily evaluated. To obtain a form-factor approximation expand $\sum b_{nruj}(\mathbf{p}_A,\mathbf{q})t_r(\mathbf{p}_A,\mathbf{q})t_u(\mathbf{p}_A,\mathbf{q})$ in a Taylor series about an undetermined $\mathbf{p}_A=\mathbf{p}_0$ and replace the E_A and E_C appearing in the delta function by the values determined by fixing $\mathbf{p}_A$ at $\mathbf{p}_0$. Requiring that the second term in the Taylor series vanish for an S-wave deuteron yields the condition $\mathbf{p}_0=\frac{1}{2}\mathbf{q}-\frac{1}{4}\mathbf{m}-\frac{3}{4}\mathbf{k}$; the first term in the series yields the form-factor approximation

$$T_{DS0}=-2\pi i(2\pi)^{-3}\Omega\int d^3x d^3q\sum b_{nruj}(\mathbf{p}_0\mathbf{q})\exp[-i\mathbf{q}\cdot\mathbf{x}]$$
$$\times\exp[\tfrac{1}{2}i(\mathbf{k}+\mathbf{m})\cdot\mathbf{x}]\phi_n^*(\mathbf{x})t_r(\mathbf{p}_0,\mathbf{q})$$
$$\times\delta(E_q-E_k+E_C{}^0-E_A{}^0)t_u(\mathbf{p}_0,\mathbf{q})\phi_j(\mathbf{x}).\quad(54)$$

In order to see the effects of double-scattering form factors as compared to single-scattering ones, consider the simple model in which a deuteron is all S state, pion-nucleon interactions are spin-independent and take place in the S waves only, and the static limit can be taken. Then the scattering amplitudes become

$$T_{00}=2t_0\int_0^\infty dr\,j_0(br)u^2(r),$$
$$T_{DS0}=-i\pi^{-1}(kE_k\Omega)t_0^2\int_0^\infty dr\,j_0(kr)j_0(Qr)u^2(r),\quad(55)$$

where $t_0=-2\pi(kE_k\Omega)^{-1}\sin\delta_0\exp i\delta_0$, $Q=\frac{1}{2}|\mathbf{m}+\mathbf{k}|$, $b=\frac{1}{2}|\mathbf{m}-\mathbf{k}|$. At resonance the factors preceding the integrals become $-4\pi i(kE_k\Omega)^{-1}$ and $+4\pi i(kE_k\Omega)^{-1}$, respectively. Thus, at resonance the angular distribution depends on the difference of the form-factor integrals; at energies other than a resonance energy the second contribution is small compared to the first. For backward scattering $b=k$, $Q=0$, so the form factors become identical, making $d\sigma/d\Omega\rightarrow 0$ for backward scattering at resonance. If the effects of spin and isospin are taken into account in a model which differs from the previous one by having the pion-nucleon interaction take place only in the $(\frac{3}{2},\frac{3}{2})$ state, the double-scattering effect does not cancel the single scattering at resonance, but instead, decreases the cross section to a value 134/243 of its single-scattering value.

Note that the double-scattering form factor has a maximum for backward scattering, decreasing to a very small value for forward scattering. The triple-scattering form-factor amplitude for all intermediate pions on the energy shell and at S-wave resonance with the nucleons, by arguments similar to those given above, is

$$T_{TS0}=4\pi i(kE_k\Omega)^{-1}\int_0^\infty dr\, j_0{}^2(kr)j_0(br)u^2(r)\,, \qquad (56)$$

which is reminiscent of the single-scattering form, but is much smaller (it cancels the double-scattering amplitude for forward scattering).

In a nonstatic calculation it is convenient in performing the $\mathbf{q}$ integrations in (54) to divide $b_{nruj}(\mathbf{p}_0\mathbf{q})t_r(\mathbf{p}_0\mathbf{q})\times t_u(\mathbf{p}_0\mathbf{q})$ into simple angular factors rapidly varying in $\mathbf{q}$ and Lorentz transformation factors which vary slowly in $\mathbf{q}$, and to take average values of the slowly varying factors (another Taylor-series expansion, if you will). For convenience, one uses the phase shifts chosen for a given scattering angle, in the single-scattering amplitude for the double-scattering amplitude also, and approximates the pion-nucleon barycentric-scattering angles by their pion-deuteron barycentric values. The integration over $\mathbf{q}$ is then easily effected using the delta function and the spherical-harmonic decomposition of $\exp(-i\mathbf{q}\cdot\mathbf{x})$ in which l values of 0 and 2 survive; the integration over angles of $\mathbf{x}$ is effected the same way, with l values of 0, 2, 4, and 6 surviving; and finally the seventeen different radial integrals are computed numerically.[25]

The cross sections computed including the double scattering form-factor amplitude can be found in Table I. There are two calculations, one neglecting the deuteron D state in the double-scattering amplitude, one including it. Both double-scattering calculations neglect the pion-nucleon S-wave phase shifts as well as the small P-wave phase shifts.

V. OUTLOOK

One argument in favor of the single-scattering form-factor approximation is that it works for pion-laboratory energies of[4] 61 MeV and[5,11] 85 MeV with the forward choice of the spectator-nucleon momentum. The differences between the optimal choice and the forward choice and between reasonable static models and more realistic models are slight at these energies. At higher energies (142 MeV and 300 MeV) the different form-factor approximations strongly differ from one to another. Few comparisons have been carried out at 300 MeV; Table I compares calculations at 142 MeV.

The first calculation, patterned after one[12] claiming to fit the data, uses an inconsistent static model which assumes that nucleons are infinitely massive within the deuteron but which gives the deuteron its correct mass and which uses experimental phase shifts instead of experimental cross sections in the static model. The second calculation, patterned after the successful calculations at 61 and 85 MeV, uses the forward choice and ignores the pion-nucleon S-wave phase shifts, the deuteron D state, and the repulsive-core effects. Note that the first calculation also makes these assumptions for convenience although double-scattering effects have been included in previous static calculations[12] only to lead to a conclusion (violently in contradiction to that found in Sec. IV and by other authors)[16] that double scattering is negligible at all angles and energies less than 300 MeV.

The third calculation is similar to the second except that the optimal choice of the spectator-nucleon momentum is used; the marked difference between this calculation and the second one for backward scattering is a sign that the form-factor approximation is breaking down since the "choice" should not be critical. The difference is caused principally by the rapid rise of the $(\frac{3}{2},\frac{3}{2})$ phase shift in the third calculation as the scattering angle increases.

The importance of the S-wave phase shifts (actually included in the previous successful calculations at 61 and 85 MeV) can be seen in the fourth calculation for the 60° scattering where the $(\frac{3}{2},\frac{3}{2})$ phase shift has not yet achieved dominance. The depressing effect of the repulsive core is quite important for backward scattering, as the fifth calculation shows. The sixth shows that the deuteron D state reduces scattering significantly at 90° and increases it at 180°. The 20% decrease at 90° is produced by a 6% D state. Preliminary computations indicate that the D-state effect at 300 MeV would be comparable to the S-state effect (if a form-factor approximation holds). In all these calculations only the single-scattering form-factor approximation (51) has been used.

The seventh calculation shows the results of including a form-factor double-scattering amplitude computed for a pure S-state deuteron; the eighth shows the result of adding the D state to that amplitude. As was seen qualitatively in Sec. IV, the double scattering reduces the backward scattering very much; but not enough to agree with experiment,[7,18] given as the ninth cross section of Table I.

One reason that the theoretical calculations are too large in the backward direction may be that the form-factor approximation overestimates the scattering amplitude by extracting a constant pion-nucleon amplitude from the simple-impulse approximation integral [Eq. (45)]. The criterion for the validity of the form-factor approximation is that the extracted functions vary slowly as a function of spectator nucleon momentum compared to the wave-function factor. Near the pion-nucleon resonance one can estimate a

[25] More details of the double scattering calculation can be found in the author's Ph.D. dissertation.

"width" for the extracted function of 450 MeV/c obtained by multiplying the usual width by the mass ratio $(E_k+M)/E_k$. On the other hand, the "width" of the wave-function part can be estimated by performing the $\mathbf{x}$ integrals for the special case of the asymptotic-deuteron wave function; the resulting function is

$$F(\mathbf{p}_B)=8\pi\alpha[\alpha^2+(\mathbf{p}_B+\tfrac{1}{2}\mathbf{m})^2]^{-1}[\alpha^2+(\mathbf{p}_B+\tfrac{1}{2}\mathbf{k})^2]^{-1} \quad (57)$$

whose "width" for backward scattering is 250 MeV/c, which is not very small compared to 450 MeV/c.

If this explanation of the backward-scattering discrepancy is valid it suggests that the pion-deuteron scattering amplitude depends critically on the pion-nucleon amplitudes for many values of the scattering parameters, most of which are off the energy shell. In that case a correct calculation of the pion-deuteron scattering amplitude awaits the development of methods adequate to handle the unphysical values of the pion-nucleon amplitudes.[26] Pion-deuteron scattering will then provide information about features of the pion-nucleon interaction not accessible to pion-nucleon scattering experiments as well as information about the deuteron wave function.

ACKNOWLEDGMENTS

For the suggestion that the impulse approximation be derived using the Heitler-London method and for many enlightening discussions concerning all sections of this work the author is indebted to Professor R. E. Cutkosky. Helpful discussions with Dr. T. Fields, Dr. E. G. Pewitt, Dr. S. S. Schweber, Dr. G. Stranahan, Dr. G. B. Yodh, and Dr. H. D. Young are also gratefully acknowledged.

[26] Such methods are proposed by R. E. Cutkosky, in *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester*, edited by E. C. G. Sudershan, J. H. Tinlot, and A. C. Melissinos (Interscience Publishers, Inc., New York, 1960), pp. 236-243.

PHYSICAL REVIEW VOLUME 131, NUMBER 4 15 AUGUST 1963

Polarization of Protons from the High-Energy Photodisintegration of Deuterium*

F. J. Loeffler, T. R. Palfrey, Jr., and T. O. White, Jr.
Department of Physics, Purdue University, Lafayette, Indiana
(Received 1 April 1963)

Photons with a mean energy of 294 MeV yielded photoprotons from a deuterium target at a mean angle of 58° in the laboratory. The protons were observed in a carbon-plate spark chamber and measurements of their elastic scattering in the carbon were used to determine the polarization components of the protons. The component parallel to the photodisintegration plane was found to be consistent with a value of zero within the statistical errors of the measurement as was expected. The component of polarization perpendicular to the photodisintegration plane was observed to be small and negative, and not inconsistent with a value of zero.

INTRODUCTION AND EXPERIMENTAL PROCEDURES

PROTONS from the photodisintegration of deuterium have been observed in a carbon-plate spark chamber. Measurements of elastic scattering in the carbon plates have been used to deduce their polarization in the direction perpendicular to the production plane defined by the photon propagation vector $\mathbf{K}_\gamma$ and the proton momentum $\mathbf{p}$. As is customary we define the polarization as positive if most of the protons have their spins in the direction $\mathbf{K}_\gamma \times \mathbf{p}$.

A 320-MeV bremsstrahlung beam from the Purdue synchrotron was used in conjunction with a 2-in.-diam liquid-deuterium target. The resultant protons from elastic photodisintegration processes were observed and identified at a mean-production angle of 58° in the lab (72° in the center-of-mass system) by a four scintillation counter telescope. This detector triggered the spark chamber when traversed by protons with production energies between 155 and 180 MeV in the lab. The mean proton energy at production was 165 MeV corresponding to a photon energy of 294 MeV. Protons from inelastic processes (i.e., pion production) were excluded by kinematic limits and the estimated pion contamination was less than 5%. The spark-chamber configuration as seen by a proton entering from the direction of the target was as follows: three thin aluminum plates, each $\frac{1}{16}$ in. thick; six carbon plates, each $\frac{1}{2}$ in. thick; three thin aluminum plates ($\frac{1}{16}$ in.); two carbon plates ($\frac{1}{2}$ in.); two thin aluminum plates ($\frac{1}{16}$ in.). The lateral dimensions of all plates were 8 in.×16 in. Figure 1 shows the experimental arrangement.

Approximately 75 000 photographs were taken using a 90° stereo, Fresnel mirror system, and at least 98% of these showed a single track coming from the target direction.

Scanning and measuring of these photographs were

* This work was supported in part by the U. S. Atomic Energy Commission.